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A rotating dust cloud in general relativity

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Abstract. An axially symmetric, stationary exact solution of Einstein's equations for dust is studied. It is asymptotically flat, and represents a rotating dust cloud extending tenuously to infinity, containing a singularity at the centre. An explanation is given as to why there exists no corresponding solution in Newtonian theory.

1. Introduction

Consider steady, axially symmetric motion of dust in which the particles rotate in circles about an axis (Oz). In Newtonian mechanics this implies that there is no density gradient in the z direction, because if there were, gravity would make matter move parallel to Oz. In other words Newtonian theory does not allow an isolated, axially symmetric, steadily rotating dust cloud.

This is easily seen mathematically as follows. Let the velocity of a dust particle P be given by:

$$\boldsymbol{u} = \omega \hat{\boldsymbol{k}} \times \boldsymbol{R}, \tag{1.1}$$

where $\boldsymbol{R} = \overrightarrow{\mathrm{OP}}$, $\hat{\boldsymbol{k}}$ is the unit vector parallel to Oz and ω is a function of z and r, r being the perpendicular distance of P from Oz. The equations of motion and of continuity for steady motion are:

$$(\boldsymbol{u} \,.\, \nabla)\boldsymbol{u} = -\nabla \psi, \tag{1.2}$$

$$\nabla \,.\, (\rho \boldsymbol{u}) = 0, \tag{1.3}$$

where ψ is the gravitational potential satisfying:

$$\nabla^2 \psi = 4\pi\rho, \tag{1.4}$$

ρ being the density. The z component of (1.2) gives, because of (1.1), $\partial\psi/\partial z = 0$, so differentiating (1.4) with respect to z we find:

$$\partial\rho/\partial z = 0, \tag{1.5}$$

and the density gradient parallel to Oz must vanish. Furthermore, because of the axial symmetry, (1.2) reduces to the single equation:

$$\omega^2 r = \partial\psi/\partial r, \tag{1.6}$$

so, since ψ is independent of z, so is ω. Hence the motion is the same in every plane z = constant. We may summarise this by saying that *steady, axially symmetric motion of dust is necessarily cylindrically symmetric according to Newtonian theory.*

It was a surprise to me to discover that *this result is not true in general relativity* (GR). Consider a massive body with centre O, rotating about Oz: then, according to GR, it is possible for a test particle P to describe a circle in a plane perpendicular to Oz, with centre C on Oz but different from O. The spin of the central body causes a force to act on P parallel to the z axis which can balance the component of gravitational attraction in that direction. This phenomenon allows the possibility of density gradients in steadily spinning dust, and even, as we shall see, a spinning dust cloud with asymptotic flatness (though having a singularity at the centre).

The work was stimulated by the remarkable paper of Winicour (1975) giving the general solution of Einstein's equations for stationary, axially symmetric dust. Having checked Winicour's calculations (and found them correct!) I nevertheless decided that for the purposes of this investigation it was sufficient to use the special solution of van Stockum (1937) discovered long ago.

The plan of the paper is as follows. In § 2 I discuss further the motion of test particles in the field of a spinning mass, in § 3 I give a solution of the van Stockum class which refers to a spinning dust cloud. The paper ends with a conclusion in § 4.

2. Closed test particle orbits in the field of a spinning body

We are interested in circular orbits whose centres lie on the axis of the body's spin but are different from the centre of the body, and whose planes are perpendicular to this axis. For brevity we shall refer to these as *non-equatorial circular orbits*. We shall use the Kerr solution (Misner *et al* 1973);

$$ds^2=\Sigma^{-1}\Delta(dt-a\sin^2\theta\,d\phi)^2-\Sigma^{-1}\sin^2\theta[(l^2+a^2)\,d\phi-a\,dt]^2-\Sigma\Delta^{-1}\,dl^2-\Sigma\,d\theta^2, \tag{2.1}$$

where

$$\Delta=l^2-2ml+a^2, \qquad \Sigma=l^2+a^2\cos^2\theta.$$

First we consider the approximation to (2.1) containing terms up to order one in m (the mass) and in a (the angular momentum per unit mass):

$$ds^2=-(1+2ml^{-1})\,dl^2-l^2(d\theta^2+\sin^2\theta\,d\phi^2)+4mal^{-1}\sin^2\theta\,d\phi\,dt+(1-2ml^{-1})\,dt^2. \tag{2.2}$$

This, being a standard solution of the linear approximation to the vacuum equations in GR, will apply to any stationary, rotating spherical mass, not only to the Kerr source, whatever that may be. It is easy to show that the world-line:

$$l=l_0, \qquad \theta=\theta_0, \qquad \phi=4mal_0^{-3}[1-(2m/l_0)]^{-1/2}s, \qquad t=[1-(2m/l_0)]^{-1/2}s, \tag{2.3}$$

is a geodesic relative to (2.2), subject to the condition:

$$\sin^2\theta_0=l_0^3(24ma^2)^{-1}, \tag{2.4}$$

between the constants l_0 and θ_0. *The test particle on this world-line describes a circle, of coordinate radius* $l_0\sin\theta_0$, *centre on the spin axis* $\theta=0$, *and in a plane perpendicular to this axis at a coordinate distance* $l_0\cos\theta_0$ *from the centre of the spinning mass*. This is a non-equatorial circular orbit.

From (2.4) we require

$$l_0^3(24ma^2)^{-1}\leqslant 1. \tag{2.5}$$

Introducing units of customary dimensions, this is:

$$l_0^3 c^4 (24mGk^4\omega^2)^{-1} \leq 1, \tag{2.6}$$

where k is the radius of gyration of the spinning mass about its axis, ω its angular velocity and G the constant of gravitation. Considering only orbits outside the Schwarzschild radius, so that $2Gm < l_0 c^2$, we find that (2.6) requires:

$$c^2(l_0\omega)^{-2} < 12k^4 l_0^{-4}.$$

Taking $k \sim l_0 \sim d$, the radius of the body, this becomes:

$$(\omega d/c)^2 > 1/12. \tag{2.7}$$

The condition (2.4) requires that the body be spinning rapidly, but is not impossible to satisfy for certain θ_0. However, it appears that the condition is not capable of satisfaction for arbitrary small θ_0.

In the Kerr solution (2.1) non-equatorial circles can be exact orbits of test particles. This can be seen if one takes the integrals of geodesic motion (Bičák and Stucklík 1976, de Felice and Calvani 1972) and requires l = constant, θ = constant, $ds^2 > 0$. One finds, contrary to the results of Wilkins (1972), that solutions exist for certain ranges of the constants of the motion. These orbits are not stable.

In this section we have shown that a non-Newtonian force, arising from the spin $\boldsymbol{h}$ of the central body, permits non-equatorial circular orbits. To first order in $\boldsymbol{h}$, the force on a test particle of mass M is

$$6GM\boldsymbol{h}c^{-2}l^{-2}\Omega \sin^2\theta \cos\theta,$$

Ω being the test particle's orbital angular velocity. This force is the reason for the existence of the rotating dust solution given in § 3. Since, as stated above, these orbits are unstable, the space–time of the rotating dust solutions may also be unstable, but we do not investigate this here.

3. Exact solution for a spinning dust cloud

Consider Einstein's equations for dust of density ρ and four-velocity u_i,

$$R_{ik} - \tfrac{1}{2}g_{ik}R = -8\pi\rho u_i u_k. \tag{3.1}$$

Van Stockum (1937) obtained the following class of exact, stationary, axially symmetric solutions:

$$ds^2 = -e^{\mu}(dz^2 + dr^2) - r^2\, d\phi^2 + (dt - n\, d\phi)^2 \tag{3.2}$$

where:

$$n = r\xi_2, \qquad \nabla^2\xi = 0, \tag{3.3}$$

$$\mu_1 = -r^{-1}n_1 n_2, \qquad \mu_2 = \tfrac{1}{2}r^{-1}(n_1^2 - n_2^2), \tag{3.4}$$

$$8\pi\rho = r^{-2}\, e^{-\mu}(n_1^2 + n_2^2), \qquad u^i = \delta^i_4. \tag{3.5}$$

Here suffixes 1 and 2 mean partial differentiation with respect to z and r respectively, and ∇^2 denotes the Laplacian operator in Euclidean three-space. The compatibility of (3.4) is ensured if (3.3) is satisfied.

These solutions form a subset of the set of stationary axially symmetric dust space–times. The subset represents *rigid rotations*: that is the shear, given in this non-expanding space–time by:

$$q_{ik} = \tfrac{1}{2}(u_{i;k} + u_{k;i}),$$

vanishes. The angular velocity:

$$w_{ik} \overset{\text{def}}{=} \tfrac{1}{2}(u_{i;k} - u_{k;i}),$$

does not vanish, and in fact satisfies

$$w^{ik}w_{ik} = 4\pi\rho,$$

as is required by Raychaudhuri's equation (Heckmann and Schucking 1962).

We shall study the special solution generated by:

$$\xi = 2hR^{-1}, \qquad R^2 = z^2 + r^2, \tag{3.6}$$

where h is an arbitrary constant. This gives:

$$n = -2hr^2R^{-3}, \tag{3.7}$$

$$\mu = \tfrac{1}{2}h^2r^2(r^2 - 8z^2)R^{-8}, \tag{3.8}$$

an additive constant in μ having been put zero because it can be removed by a scale change in the coordinates. The density is given by:

$$8\pi\rho = 4\,\mathrm{e}^{-\mu}h^2R^{-8}(4z^2 + r^2), \tag{3.9}$$

and is everywhere positive outside the singularity at $R = 0$.

As $R \to \infty$ the solution has the following properties: (i) the density tends very rapidly to zero; (ii) the metric (3.2) tends to Minkowski values; (iii) in the linear approximation (i.e. neglecting powers of h higher than the first) (3.2) reduces to (2.2) with $m = 0$, $ma = h$. Thus at infinity the gravitational field is that of a spinning body of zero mass situated at the origin.

We can also study the solution by using the locally non-rotating frame, defined by Bardeen (1970). This frame is obtained in our case by making a purely *local* transformation $\bar{\phi} = \phi + n(r^2 - n^2)^{-1}t$ which locally diagonalises the metric to:

$$\mathrm{d}s^2 = -\mathrm{e}^{\mu}(\mathrm{d}z^2 + \mathrm{d}r^2) - (r^2 - n^2)\,\mathrm{d}\bar{\phi}^2 + r^2(r^2 - n^2)^{-1}\,\mathrm{d}t^2.$$

$\sqrt{g_{44}} = r(r^2 - n^2)^{-1/2}$ is then taken as the gravitational potential and with n given by (3.7) it becomes $1 + \mathrm{O}(R^{-4})$ for large R, and contains no mass term.

The apparently zero mass is especially strange since the density is everywhere positive. One might hope to check it by calculating the mass from the invariant integral:

$$m = \int_H T^i_j \chi^j \nu_i \sqrt{(-{}^3g)}\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\phi, \tag{3.10}$$

where $\chi^i = \delta^i_4$ is the unit time-like Killing vector, ν_i is the unit normal to an infinite three-dimensional space-like hypersurface H, and ${}^3g$ denotes the three-dimensional determinant of the spatial part of the metric (Synge 1960, Cohen 1968). This, however, diverges. The most likely explanation is that one must regard the singularity at $R = 0$ as containing infinite negative mass, which is balanced by the positive mass outside.

The singularity at $R = 0$ shows curious features similar to those of the Curzon solution of the vacuum equations (Gautreau and Anderson 1967). It has a directional character: if it is approached along the lines:

$$z = kr, \qquad t = \text{constant}, \qquad r = \text{constant},$$

μ tends to $+\infty$, zero or $-\infty$ according as $8k^2 \gtreqless 1$, with the result that the density has different limits according to the path of approach to $R = 0$. This raises the question, often asked in the Curzon case, whether the singularity really represents a point source or not. A similar singularity has been noted in the case of charged rotating dust (Islam 1977).

Apart from $R = 0$ the space–time is non-singular; in particular there are no other singularities along the rotation axis $r = 0$ as one can easily see by transforming to cartesian coordinates. It can therefore be taken as representing a cloud of rotating dust, extending tenuously to infinity, and containing an isolated singularity.

4. Conclusion

We have been studying an exact stationary solution of van Stockum's class. It refers to a rigidly rotating dust cloud with an isolated singularity at its centre. The main point of interest is that it has a density gradient parallel to the rotation axis, which would not be allowed in Newtonian mechanics. The physical reason why his gradient can exist is the non-Newtonian force described in § 2.

It is an interesting question whether non-singular solutions exist for rotating dust clouds. One could construct a solution non-singular in a finite region about the origin by choosing an appropriate solution of (3.3), e.g.

$$\xi = 2z^3 - 3zr^2; \tag{4.1}$$

however, the density would tend to infinity with the distance in certain directions. Another possibility would be to use this as an interior solution only, but it does not seem possible to match it to the vacuum stationary solutions known at present. Yet another scheme would be to match the solutions given by (3.6) and (4.1), using the former as an exterior and the latter as an interior; however this turns out to be impossible.

The van Stockum class is special because its members refer to rigidly rotating dust. It would be interesting to know whether one of Winicour's more general class could be used to construct a non-singular rotating dust cloud.

References

Bardeen J M 1970 *Astrophys. J.* **162** 71
Bičák J and Stuchlík Z 1976 *Bull. Astronom. Inst. Czeckoslovakia* **27** 129
Cohen J M 1968 *J. Math. Phys.* **9** 905
de Felice F and Calvani M 1972 *Nuovo Cim.* B **10** 447
Gautreau R and Anderson J L 1967 *Phys. Lett.* **25A** 291
Heckmann O and Schucking E 1962 *Gravitation* ed. L Witten (New York: Wiley) p 438
Islam J N 1977 *Proc. R. Soc.* A **353** 523
Misner C W, Thorne K S and Wheeler J A 1973 *Gravitation* (San Francisco: Freeman) p 877
van Stockum W J 1937 *Proc. R. Soc. Edin.* **57** 135
Synge J L 1960 *Relativity: The General Theory* (Amsterdam: North-Holland) p 237
Wilkins D C 1972 *Phys. Rev.* **5** 814
Winicour J 1975 *J. Math. Phys.* **16** 1805